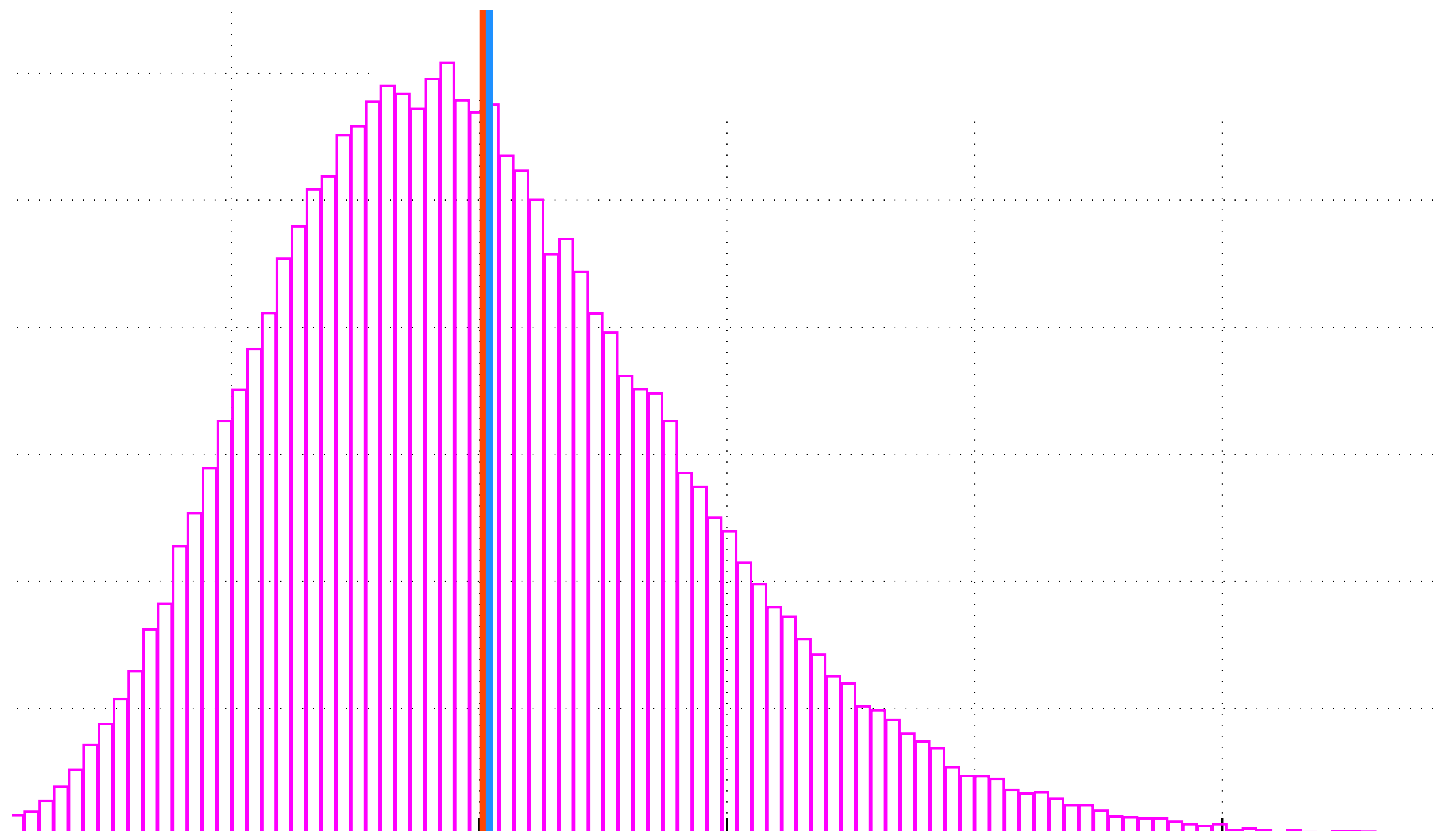
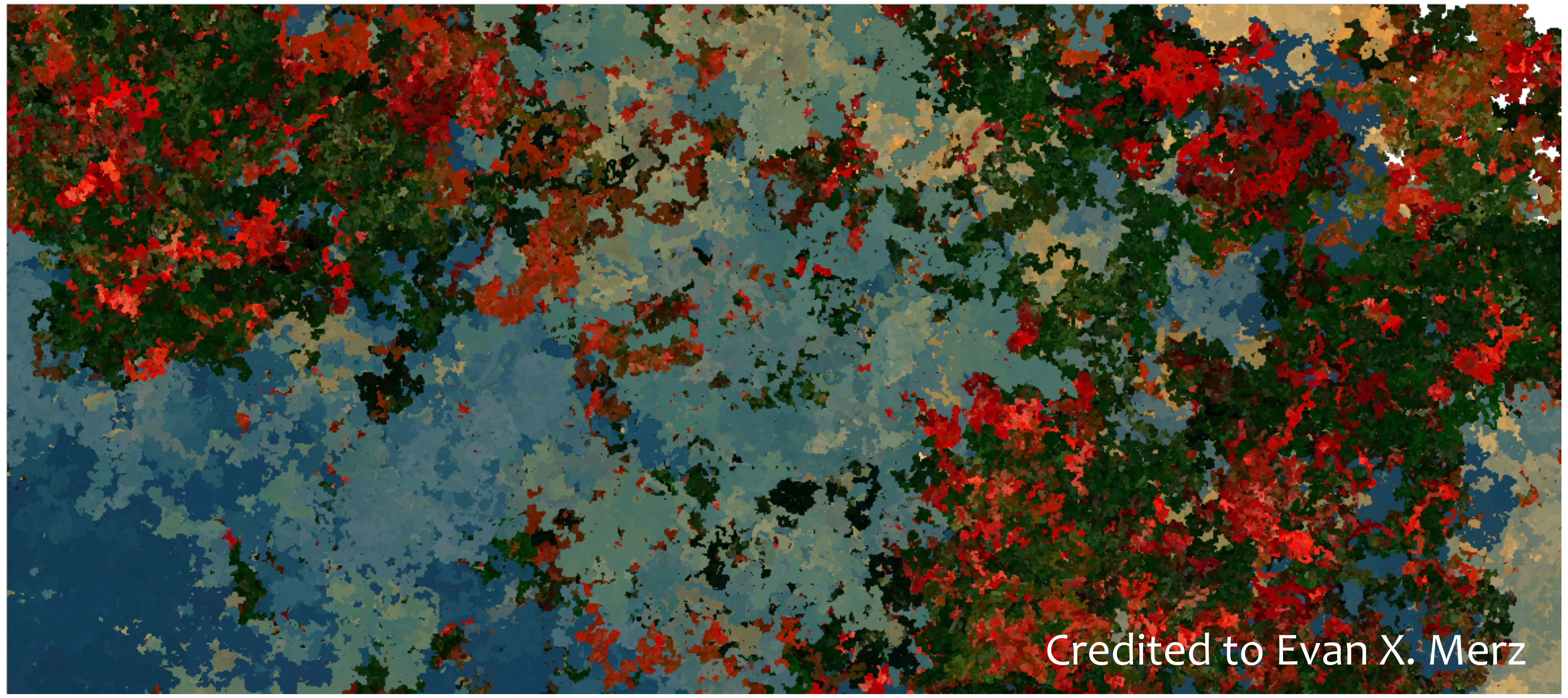


Histogram: 1D random walk



Drunk painting: 2D random walks

Bounded Expectations:
Resource Analysis for Probabilistic Programs

Van Chan Ngo   *Quentin Carbonneaux*   *Jan Hoffmann*

Carnegie Mellon University

Static resource analysis

Given: A program P

Question: What is the amount of resource as function of the inputs sizes that is required to execute P?

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Time, memory, or energy

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Techniques
Recurrence Relations
Type Systems
Abstract Interpretation
Term Rewriting
Ranking Functions
Automatic Amortized Resource Analysis

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Worst-case resource usage

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Probabilistic programs

- Are usual functional or imperative programs with two added constructs:
  - Sampling assignments to draw values at random from probability distributions, and
  - Probabilistic branchings to control program flow by observations

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Hicks 2014

"The crux of probabilistic programming is to consider normal-looking programs as if they were probability distributions"

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Hicks 2014

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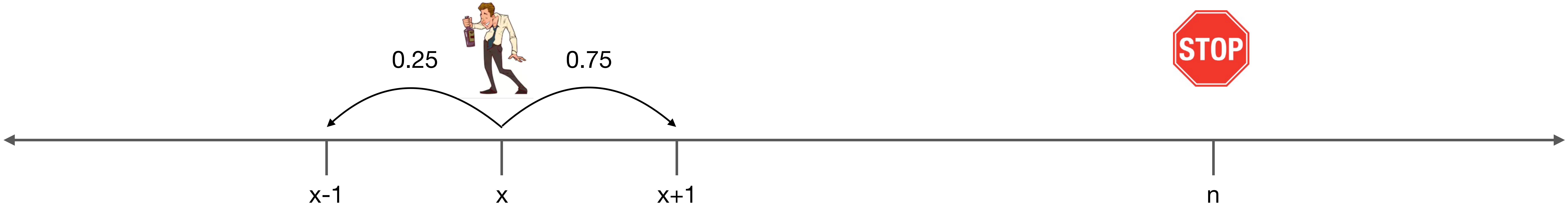
- Some probabilistic programming languages: *Probabilistic C*, *Church*, *PyMC3*, *Figaro*, *Edward*

Example: Random walk



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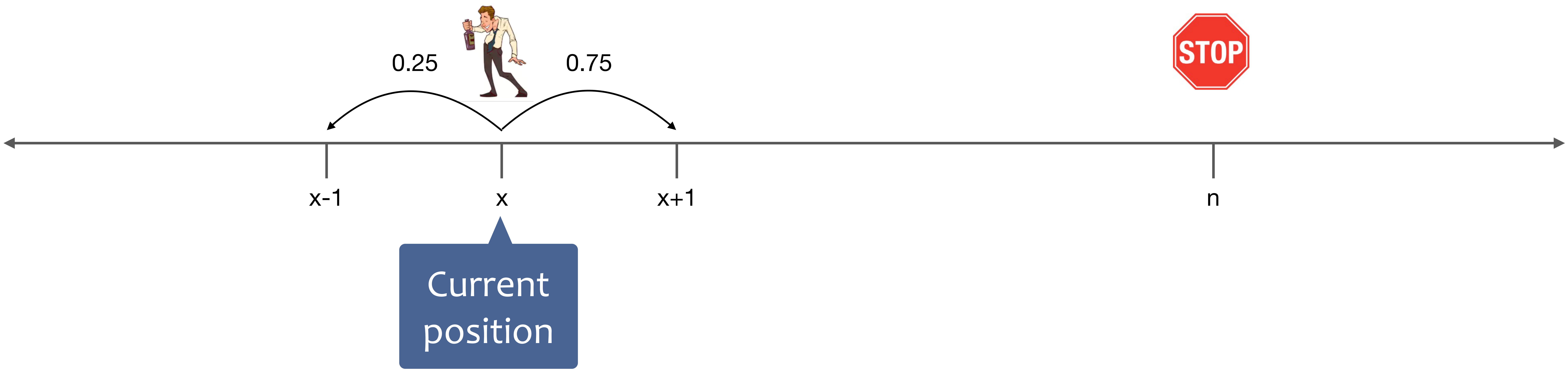

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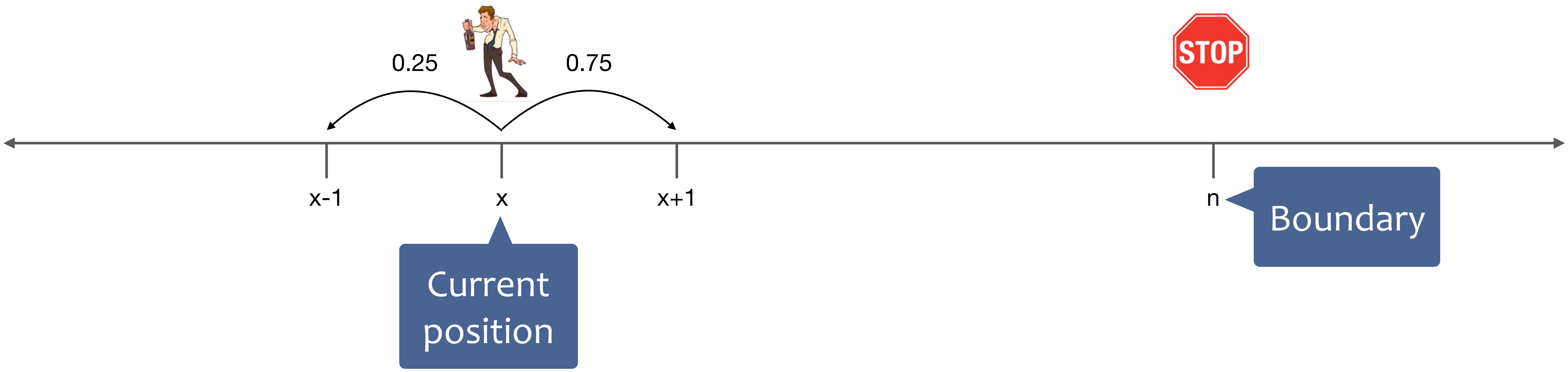

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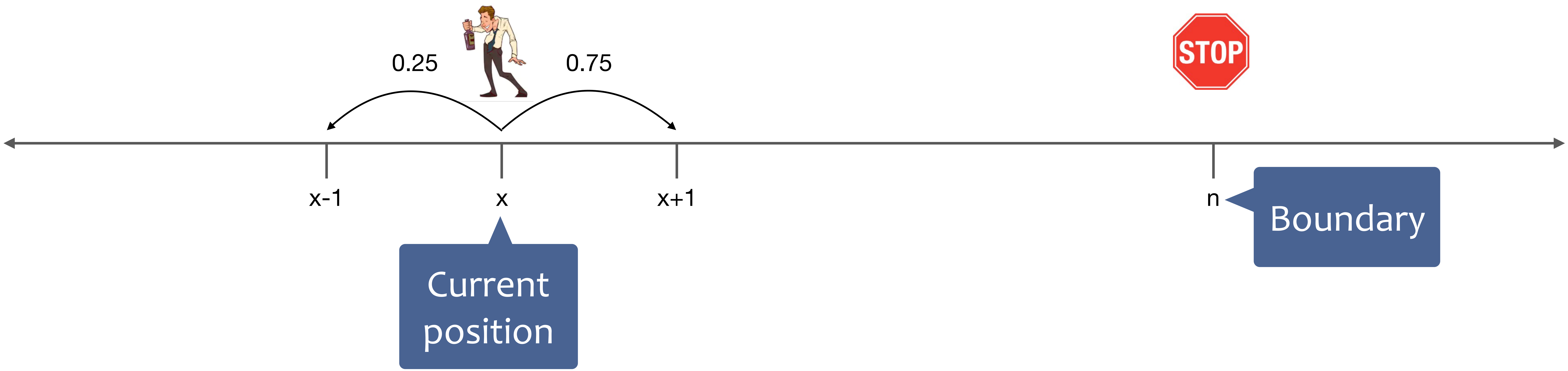


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Cost = # iterations
= walking time

Example: Random walk

Current position

Boundary

- The worst-case cost is infinite
  - It does not make sense to reason.about

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This talk: automatic bounds on the expected cost

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Why expected resource usage?

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There are many interesting applications:

- Predict the expected resource usage of sampling in probabilistic inference
- Reason about the average-case complexity of randomized algorithms, positive and almost-sure terminations

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There are many interesting applications:

- Predict the expected resource usage of sampling in probabilistic inference

- Reason about the average-case complexity of randomized algorithms, positive and almost-sure terminations

It is a technical challenge problem:

- Manual analysis is often difficult or impossible even for simple programs (e.g., requires probability theory knowledge, mathematic reasoning, …)

- No techniques that automatically infer symbolic bounds on the expected cost

Approach: Expected potential method

Kozen ('81), McIver et al ('04), Kaminski et al ('16)

Weakest Pre-expectation Calculus
Strength and conceptual simplicity
Soundness w.r.t a simple operational semantics

Hofmann and Jost ('03)

Automatic Amortized Resource Analysis
Template-based bound inference
Efficiently reduced to LP solving

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Expected Potential Method

- Automatically infers bounds on expected resource usage
- Bounds are multivariate symbolic polynomials
- Enables compositional and effective reasoning

Expected potential method

- Associate potential functions to program points
  - Function from states to non-negative values
- Potential pays the expected resource consumption and the expected potential at the following point
- The initial potential is an upper bound on the expected resource usage

$$\Phi(state) \geq 0$$

$$\Phi(state) \geq \mathbb{E}(\text{cost}) + \mathbb{E}(\Phi'(next\_state))$$

Total expectation and linearity

$$\Phi(init\_state) \geq \mathbb{E}(\Sigma\text{cost})$$

Quantitative Hoare logic

$$\{\Phi\} \quad c \ \{\Phi'\}$$

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Expected cost $\mathbb{E}(c)$

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Quantitative Hoare logic

Resource available before executing c

Expected cost $\mathbb{E}(c)$

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$\mathbb{E}(\Phi')$ is the expected resource available after executing c

For all states σ , $\Phi(\sigma)$ is sufficient to pay for the expected cost of executing c and the expected resource available after the execution w.r.t the distribution over next states

Example rules

(Q:PIF)

$$\frac{Q = p \cdot Q_1 + (1-p) \cdot Q_2 \qquad \vdash \{\Gamma; Q_1\} c_1 \{\Gamma'; Q'\} \qquad \vdash \{\Gamma; Q_2\} c_2 \{\Gamma'; Q'\}}{\vdash \{\Gamma; Q\} c_1 \oplus_p c_2 \{\Gamma'; Q'\}}$$

(Q:SAMPLE)

$$\frac{\begin{array}{cc} \Gamma \models R \in [a, b] & Q = \sum_i p_i \cdot Q_i \\ \forall v_i \in [a, b]. [\![\mu_R : v_i]\!] = p_i & \forall v_i. \vdash \{\Gamma; Q_i\} x = e \text{ bop } v_i \{\Gamma'; Q'\} \end{array}}{\vdash \{\Gamma; Q\} x = e \text{ bop } R \{\Gamma'; Q'\}}$$

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Probability that the sample value is v_i

Distribution with finite domain

Derivation: Random walk

```
while x < n:
    prob(3,1):
        x = x + 1
    else:
        x = x - 1
    tick 1
```

Derivation: Random walk

Bound on the expected cost: 2max(0,n-x) = 2|[x,n]|

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{ . ; 2|[x,n]| }
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```

Weighted sum:
3/4* (2|[x,n]| - 1) + 1/4*(2|[x,n]| + 3)

Derivation: Random walk

Bound on the expected cost: 2max(0,n-x) = 2|[x,n]|

It is the exact expected cost

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Automation

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- Fix potential functions as linear combinations of monomials with unknown coefficients

$$\Phi := \sum_i k_i \cdot m_i$$

$$M := 1 \mid x \mid M_1 \cdot M_2 \mid \max(0,\Phi)$$

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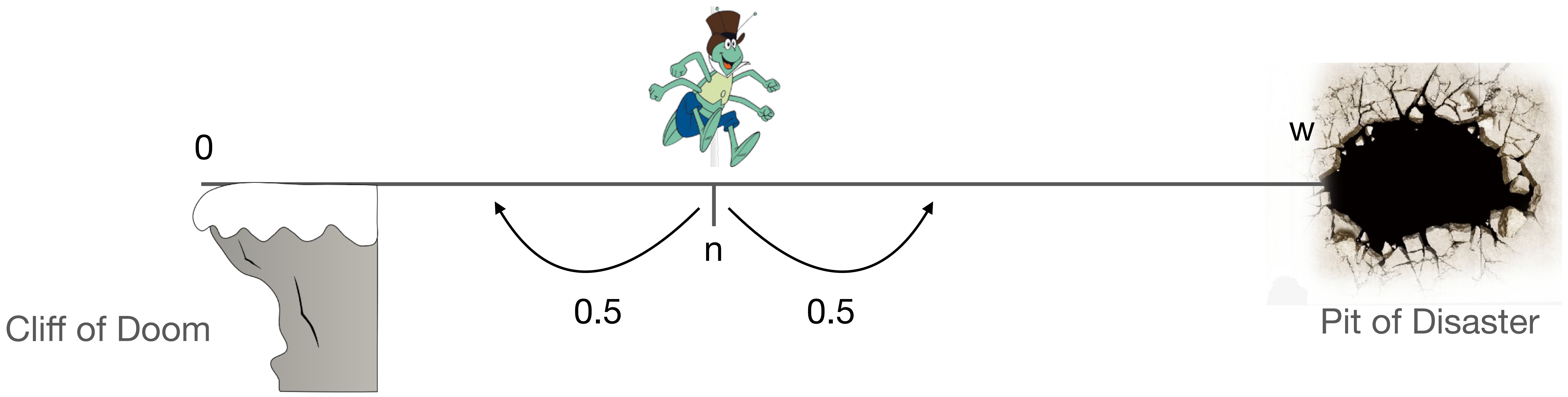
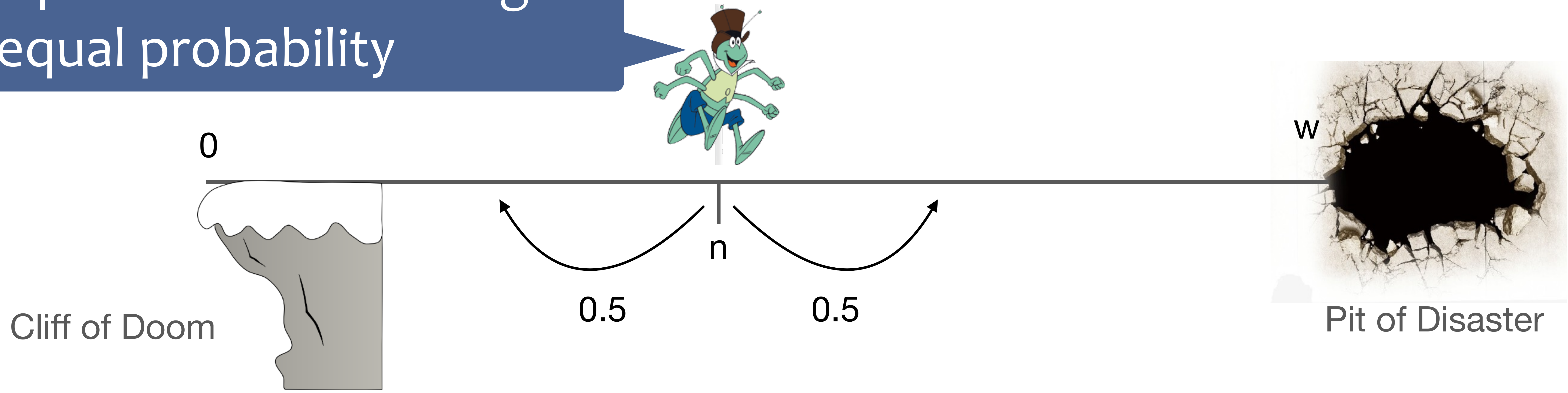
- Obtain the optimal solution by solving the generated constraints with an off-the-shelf LP solver

Example: Bug's life



Example: Bug's life

Repeatedly hops 1 inch to left or right with equal probability



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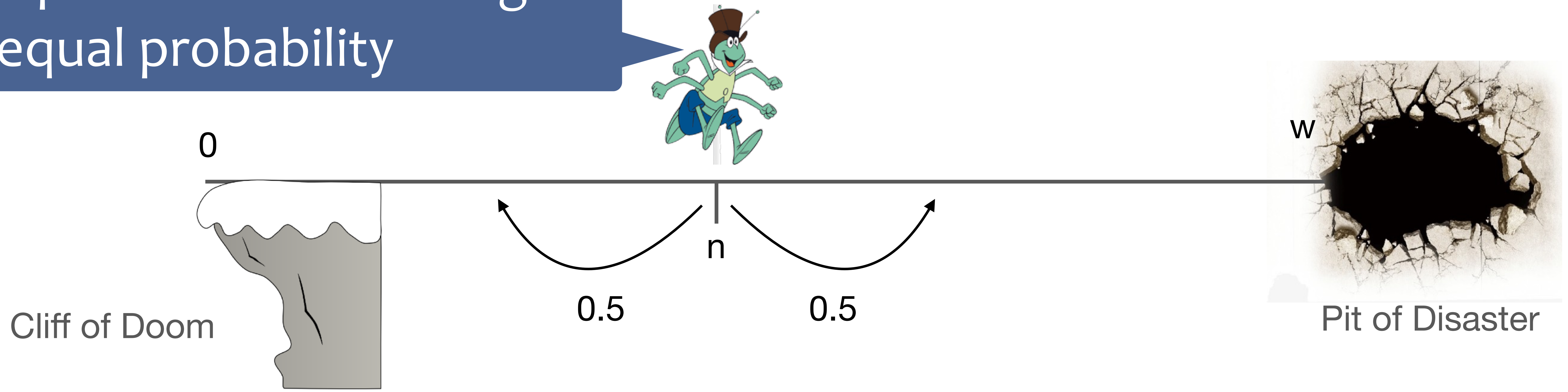


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while n > 0 && n < w:
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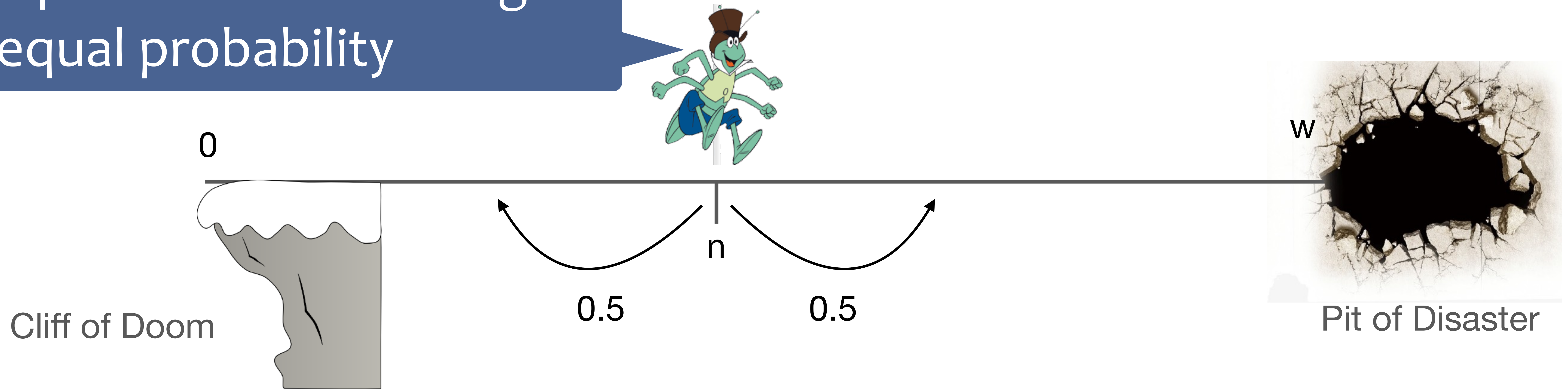


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= Bug’s life

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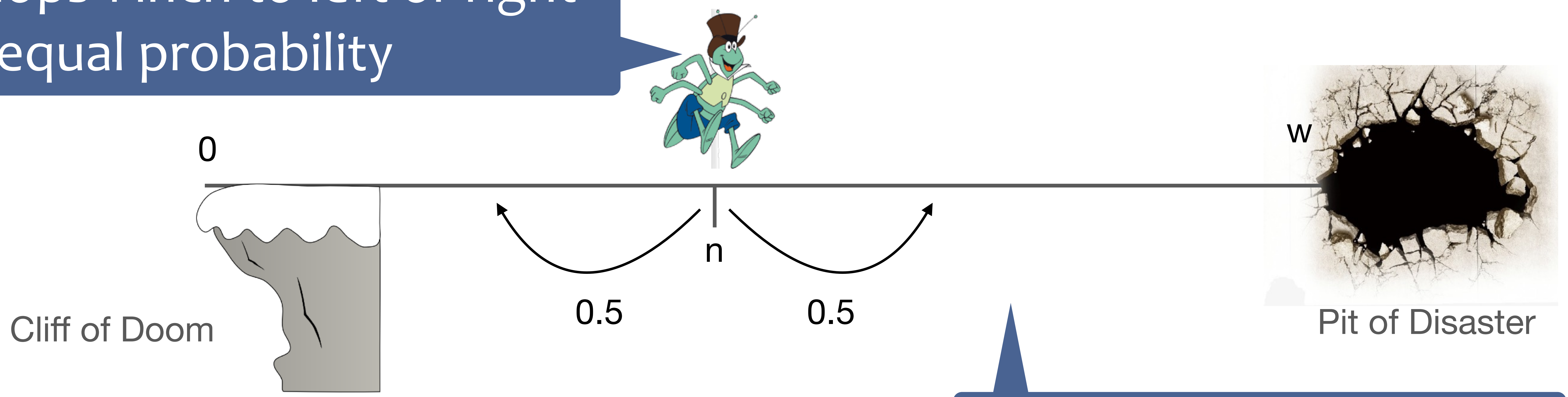

The expected life:
|[0,n]| * |[n,w]|

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Derivation: Bug's life

inv(n,w) = |[0,n]|*|[n,w]|

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{ . ; inv(n,w)}
while n > 0 && n < w:
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Derivation: Bug's life

inv(n,w) = |[0,n]|*|[n,w]|

```
{ . ; inv(n,w)}
while n > 0 && n < w:

   prob(1,1):

      n = n + 1

   else:
      { 0<n<w; inv(n,w)-|[n,w]|+|[0,n]| }
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```

Weighted sum in which terms are canceled out

Implementation: Absynth

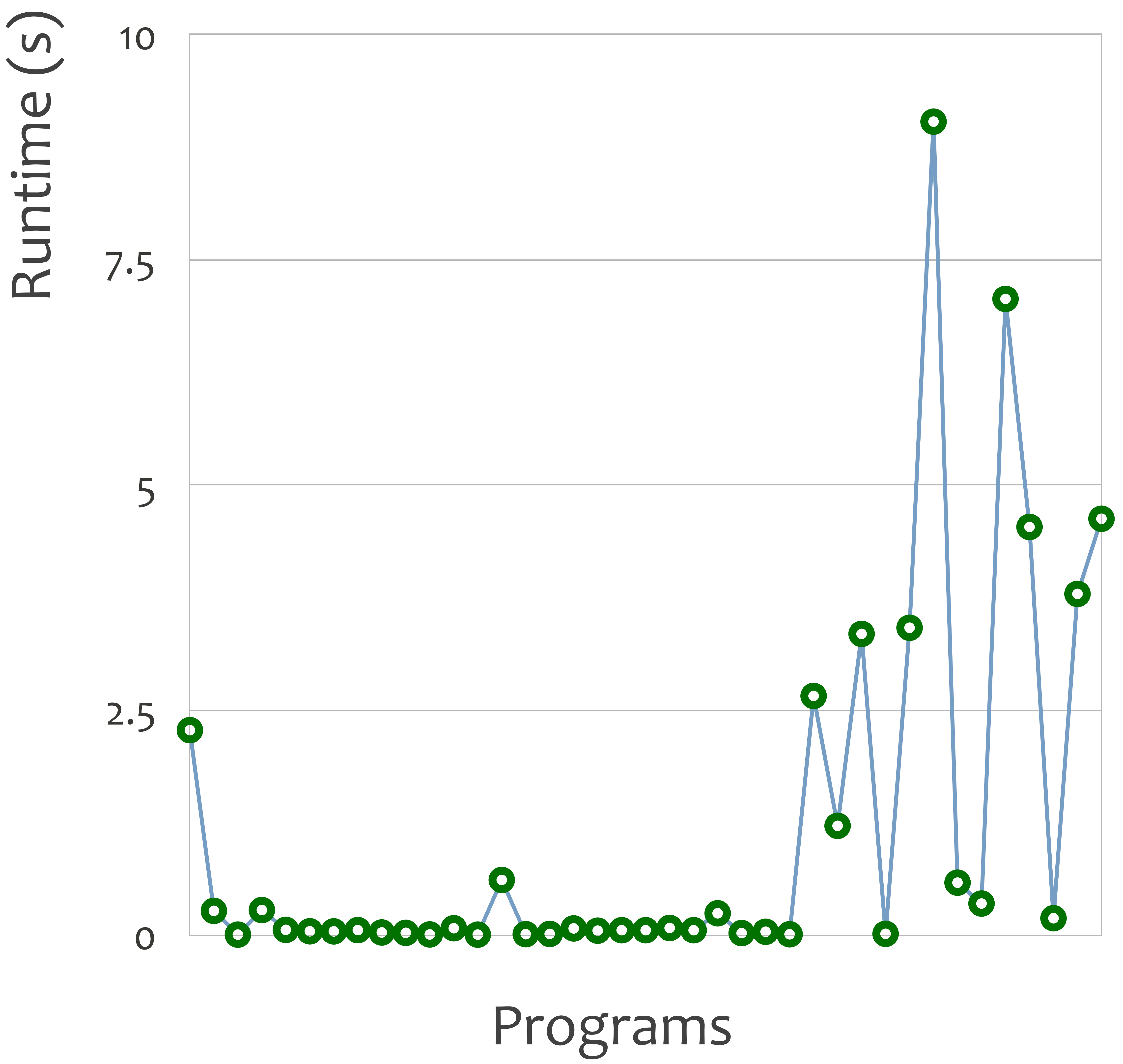
- Accepts imperative (integer) probabilistic programs

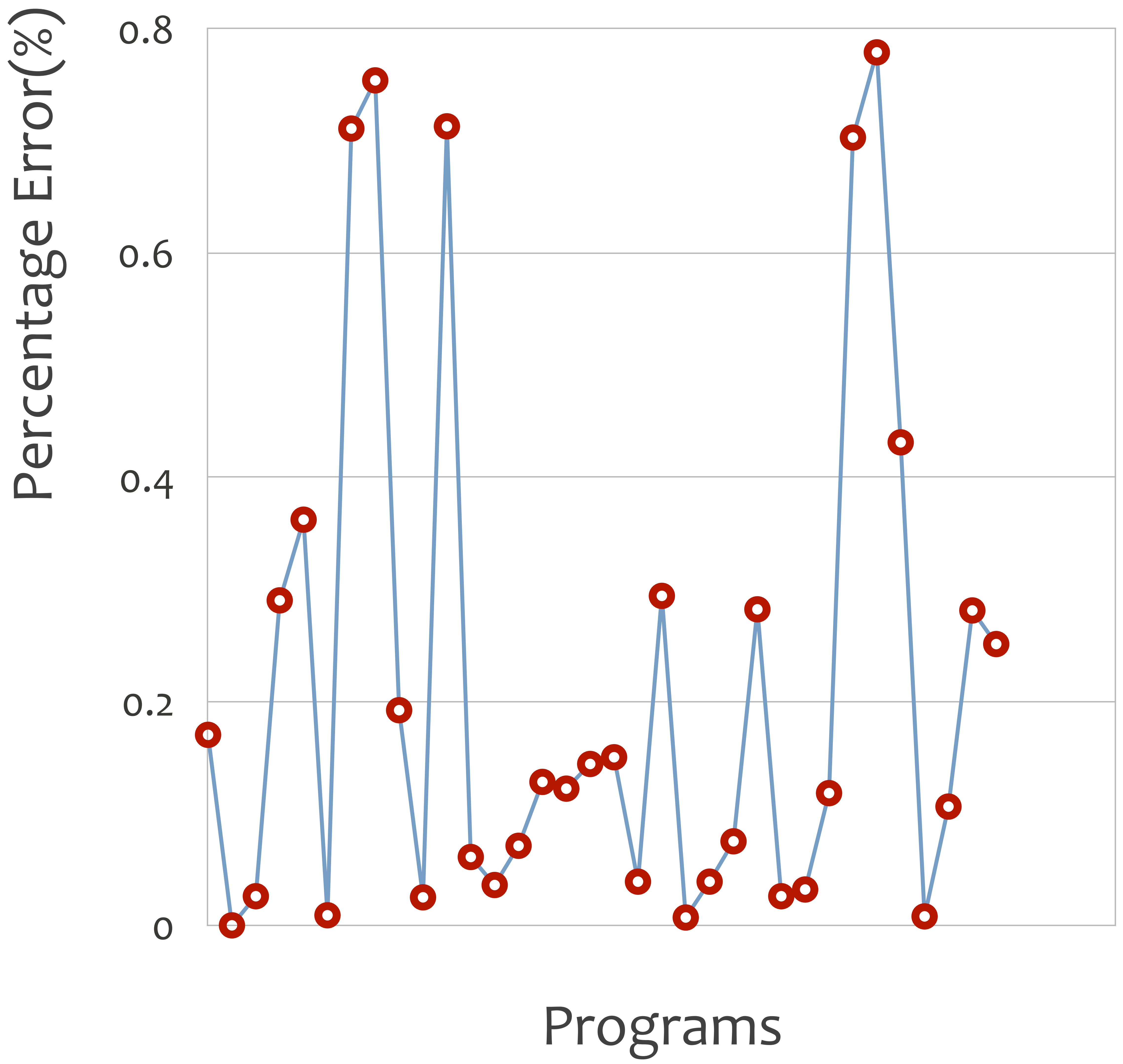
- Infers multivariate polynomial bounds on the expected resource consumption

- Automatically analyzes 40 challenging probabilistic programs and randomized algorithms with different looping patterns

- Statically derived bounds are compared with simulation-based expectations to show that constant factors are very precise

Experiments: Overview





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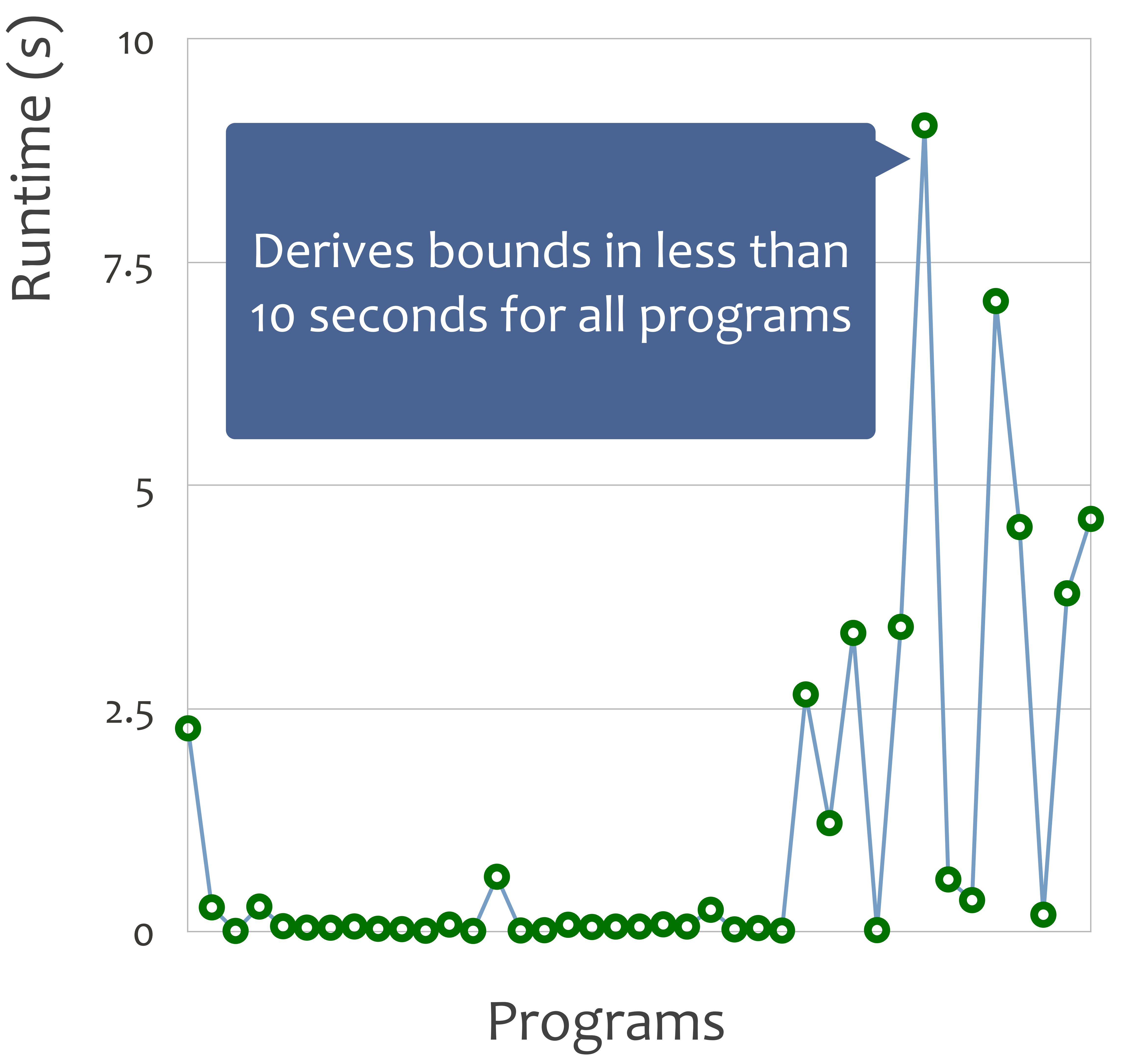

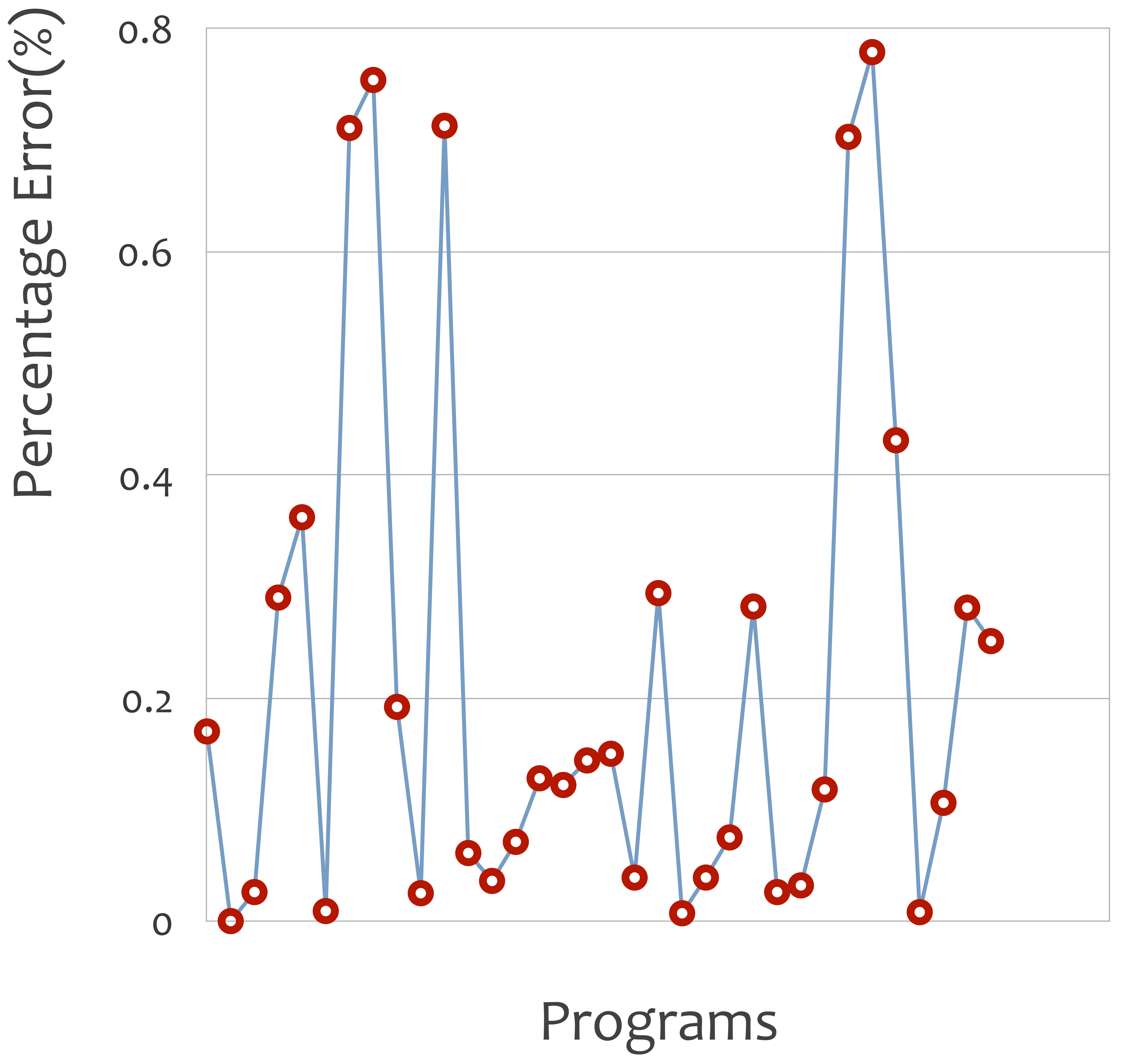

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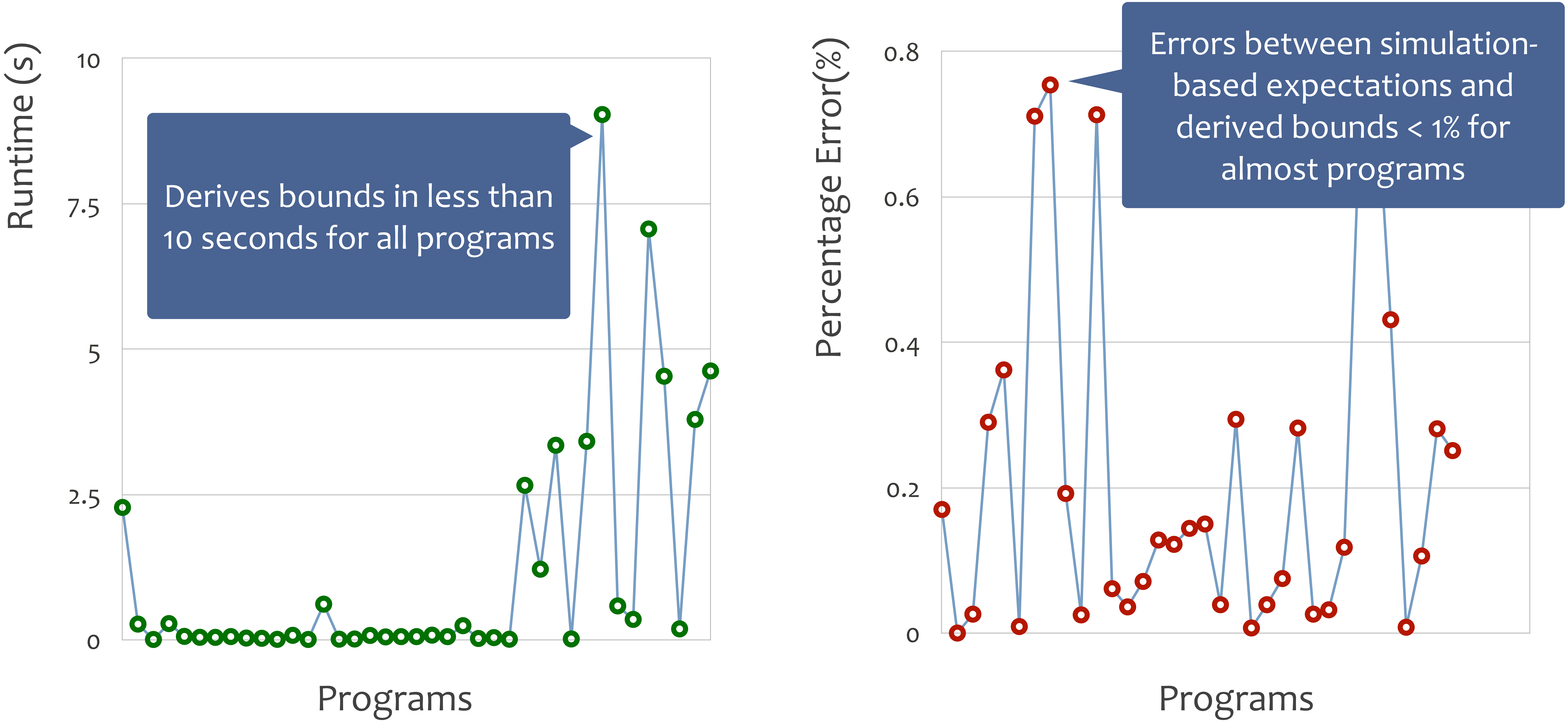
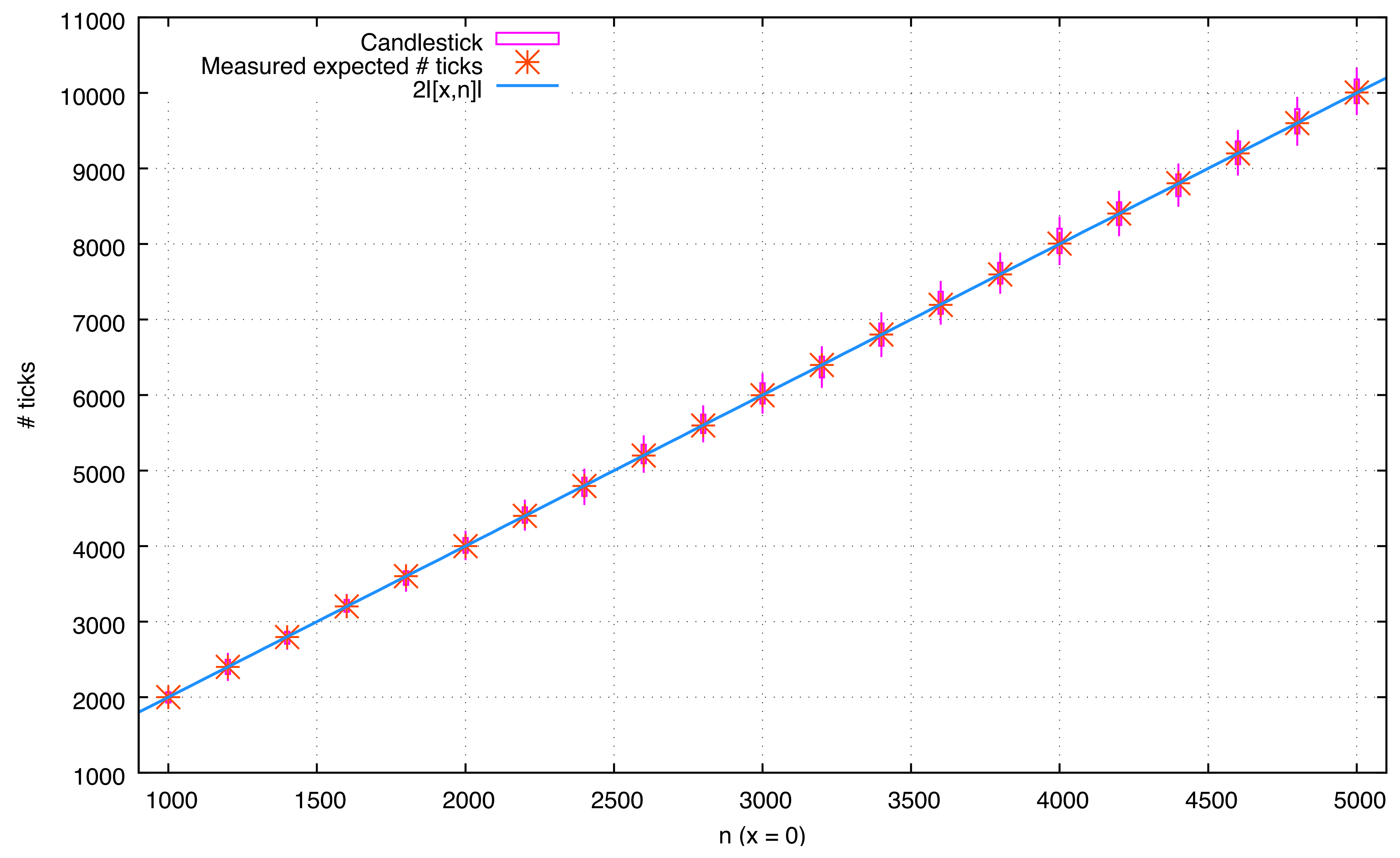
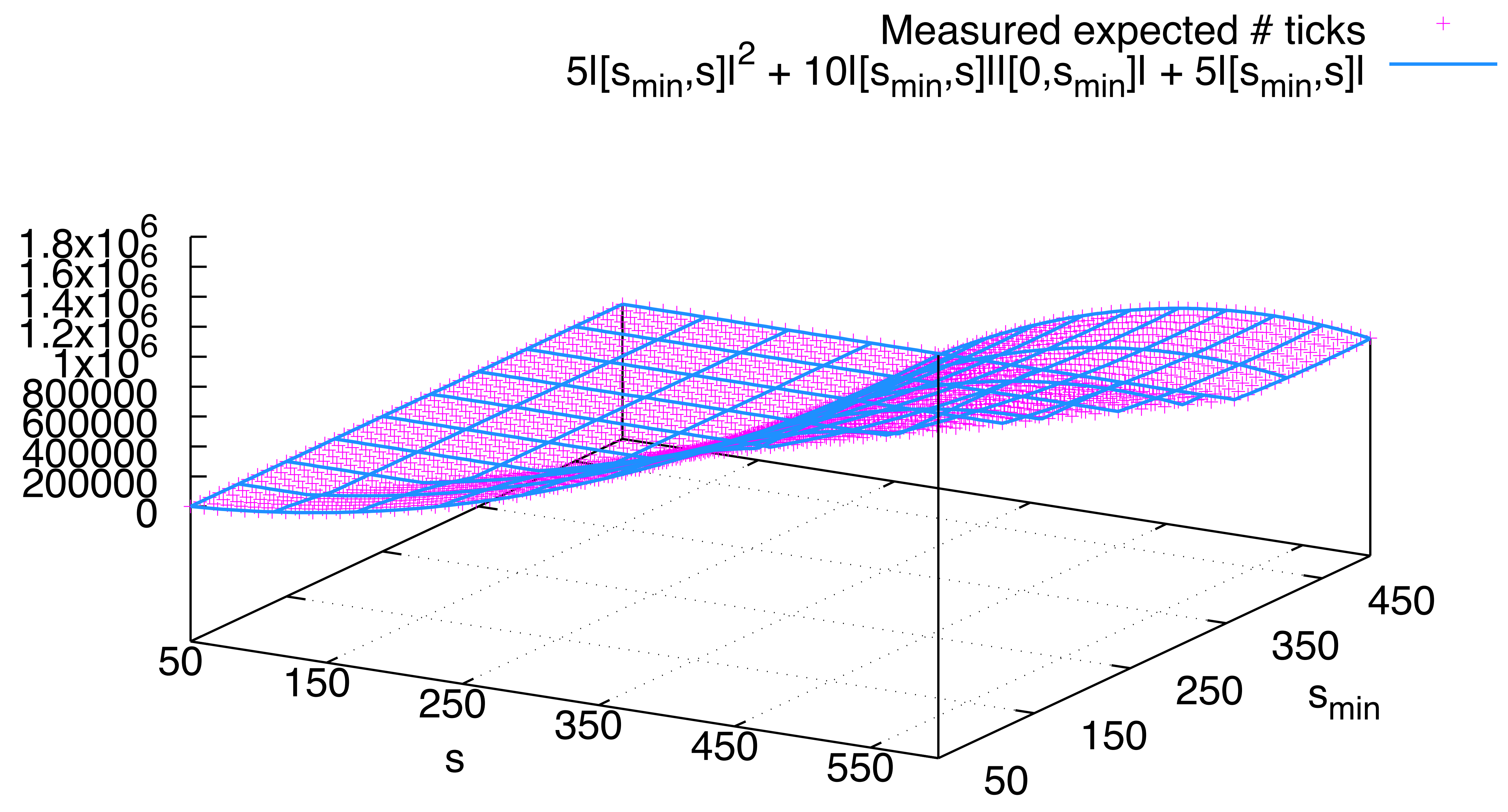


Precise constant factors

- For example, figures show the constant factors in derived bounds for random walk and polynomial programs are very precise





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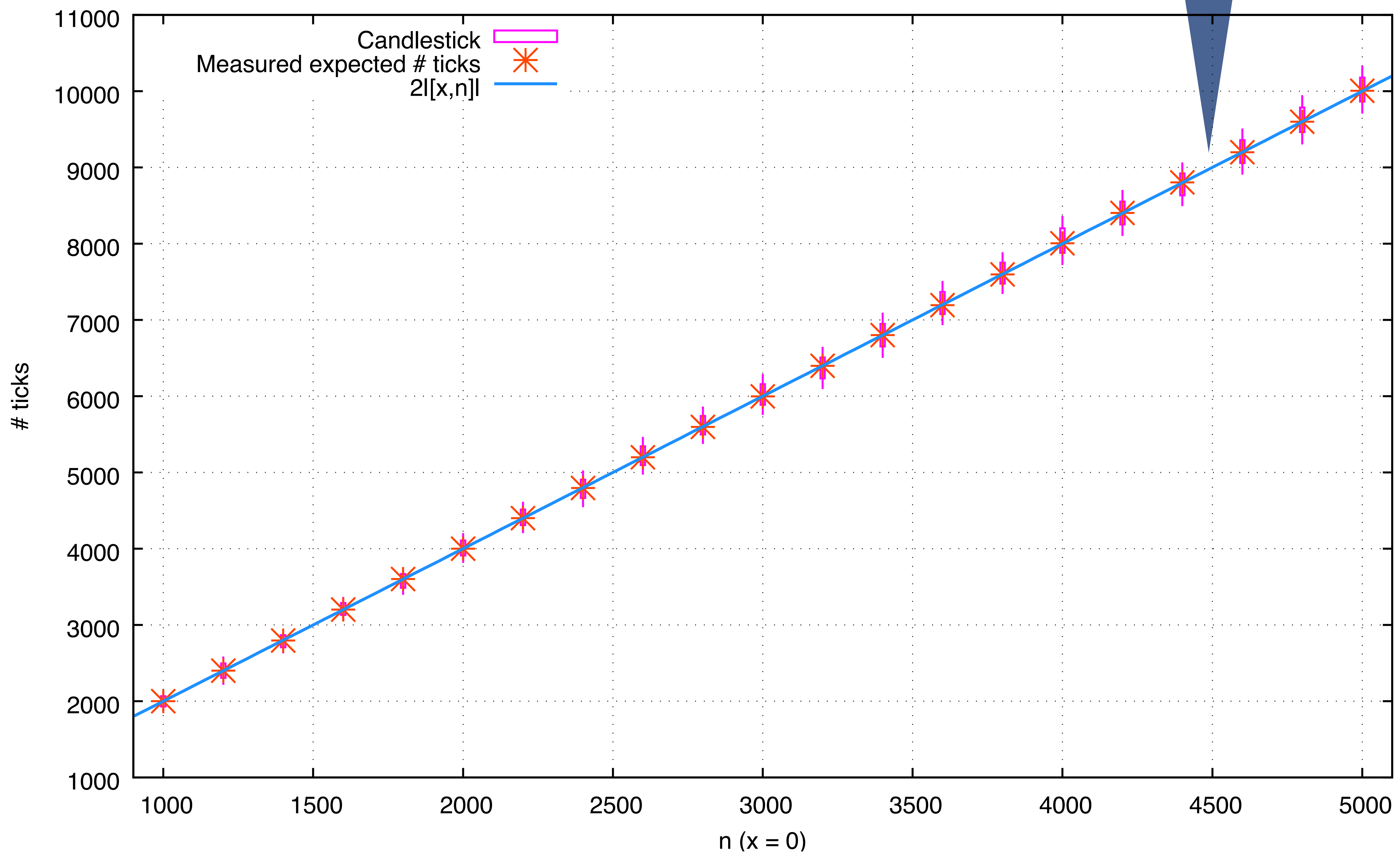
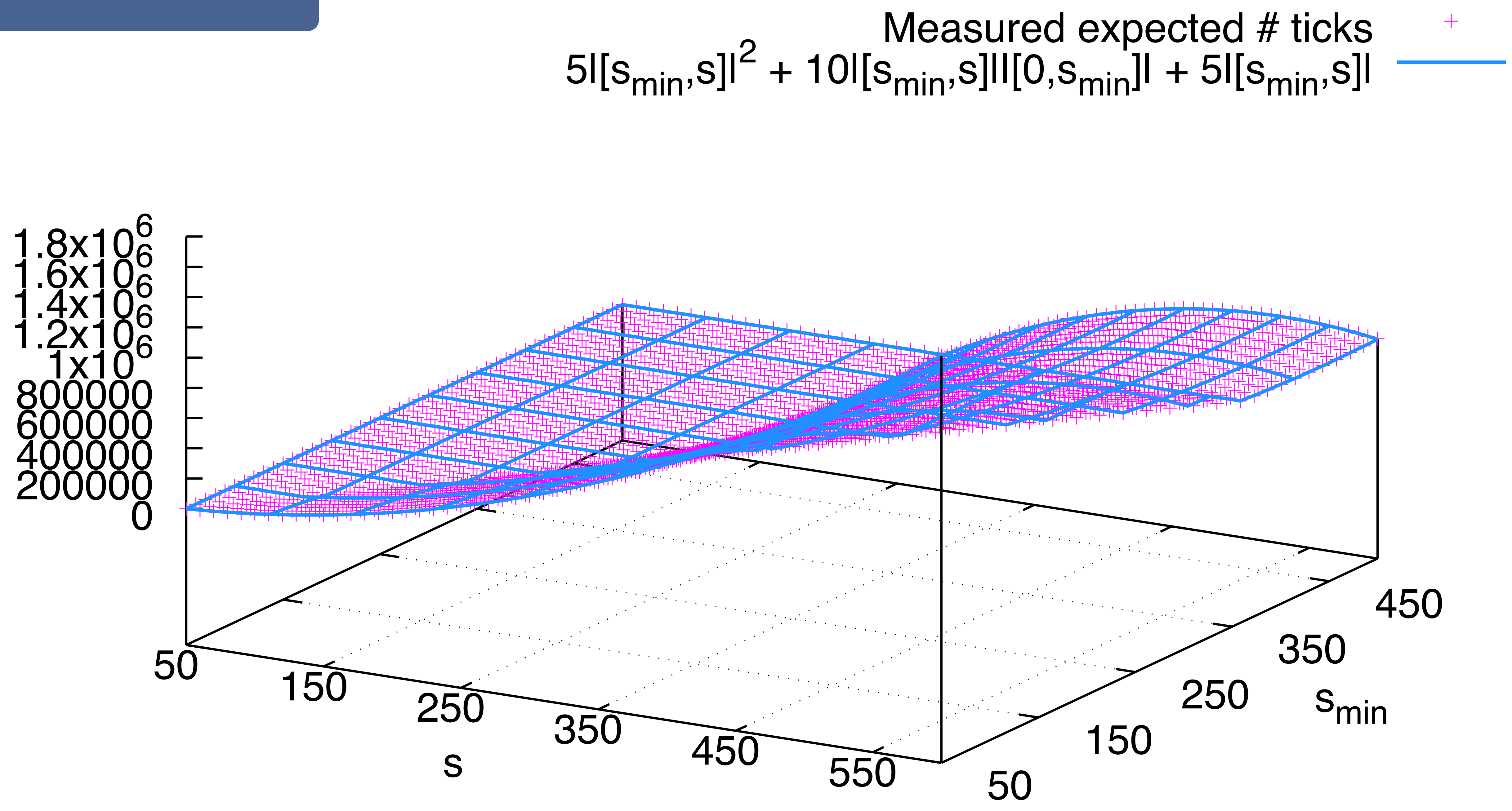
Blue lines are plotting of derived bounds





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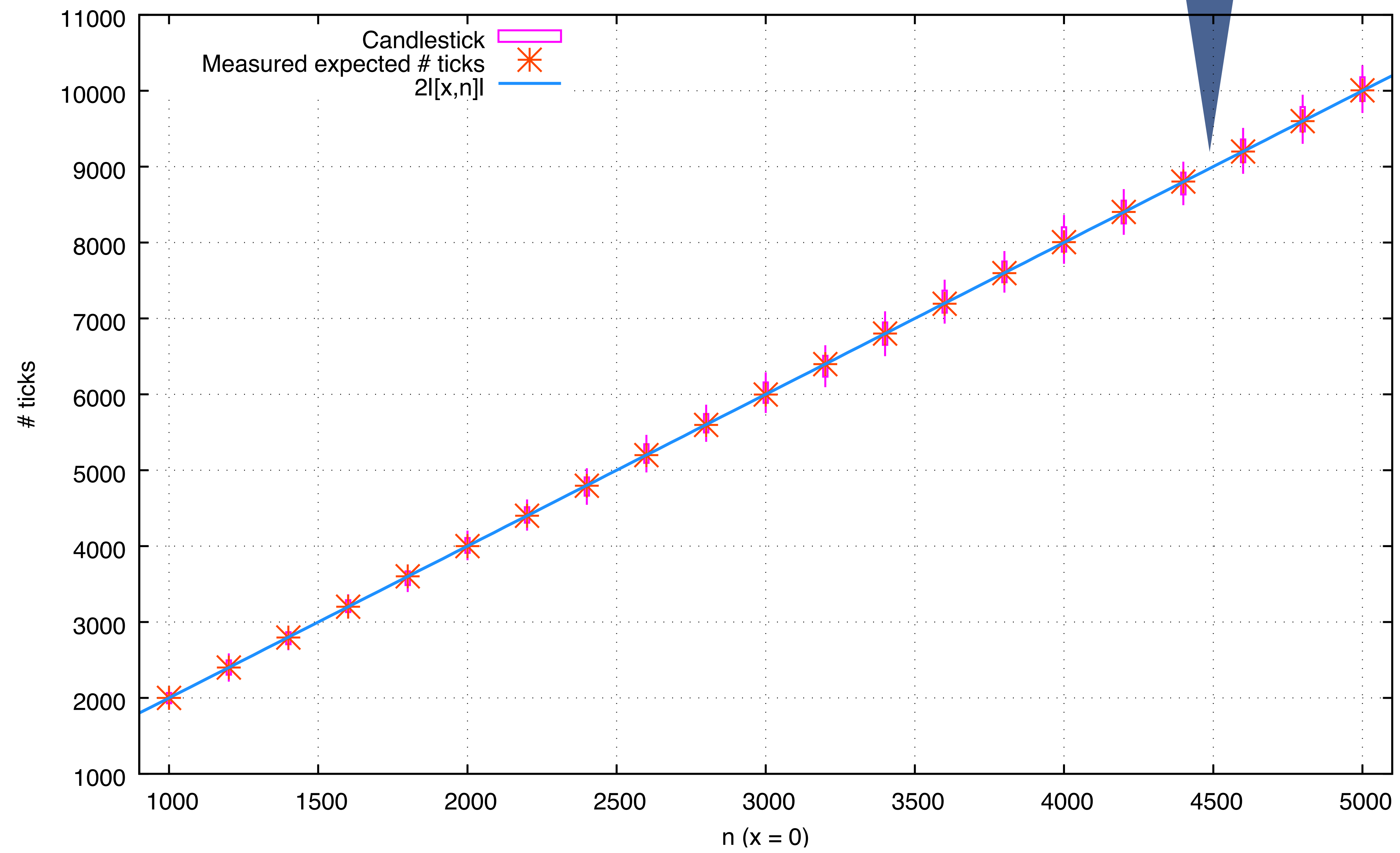
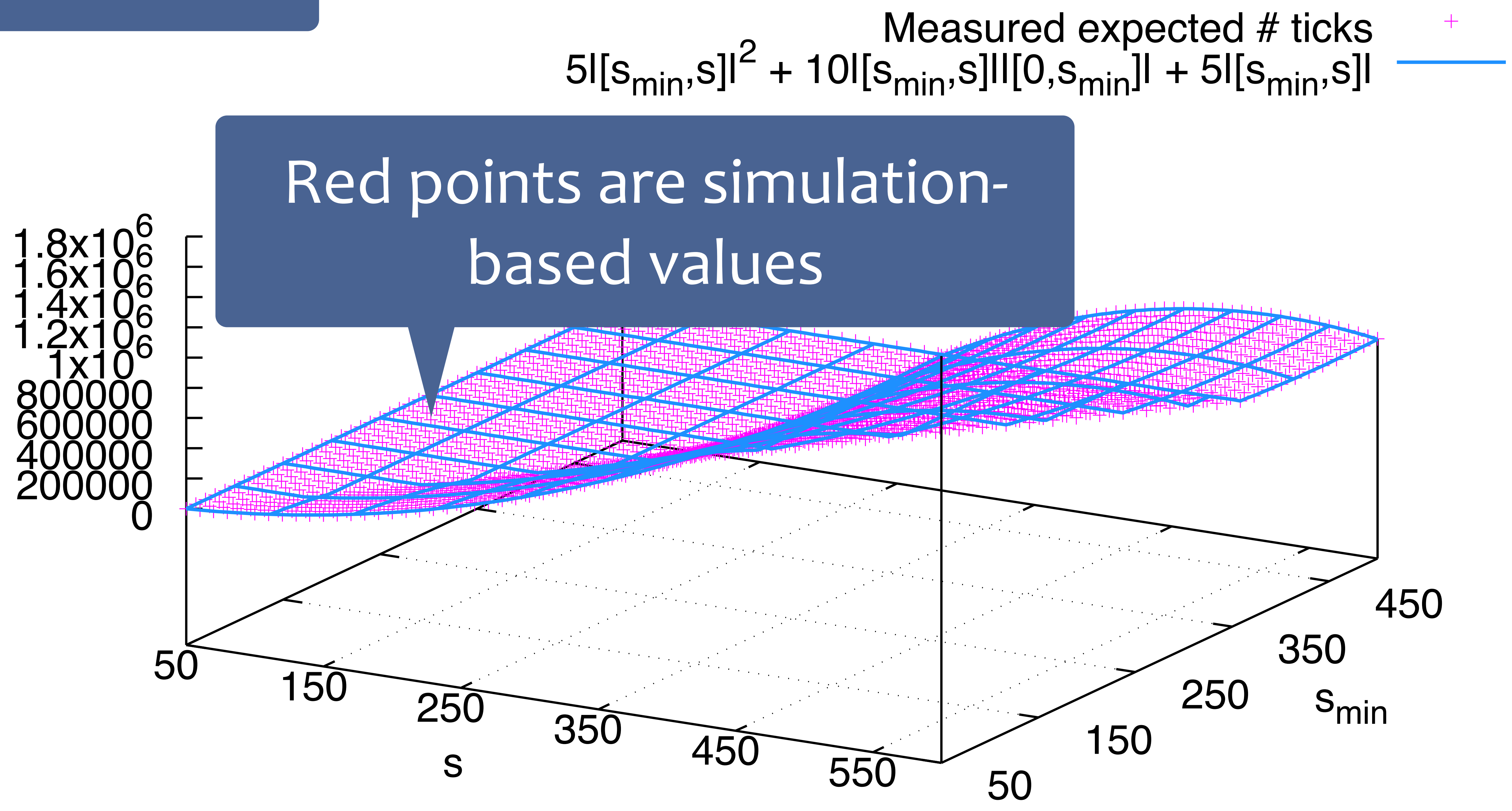
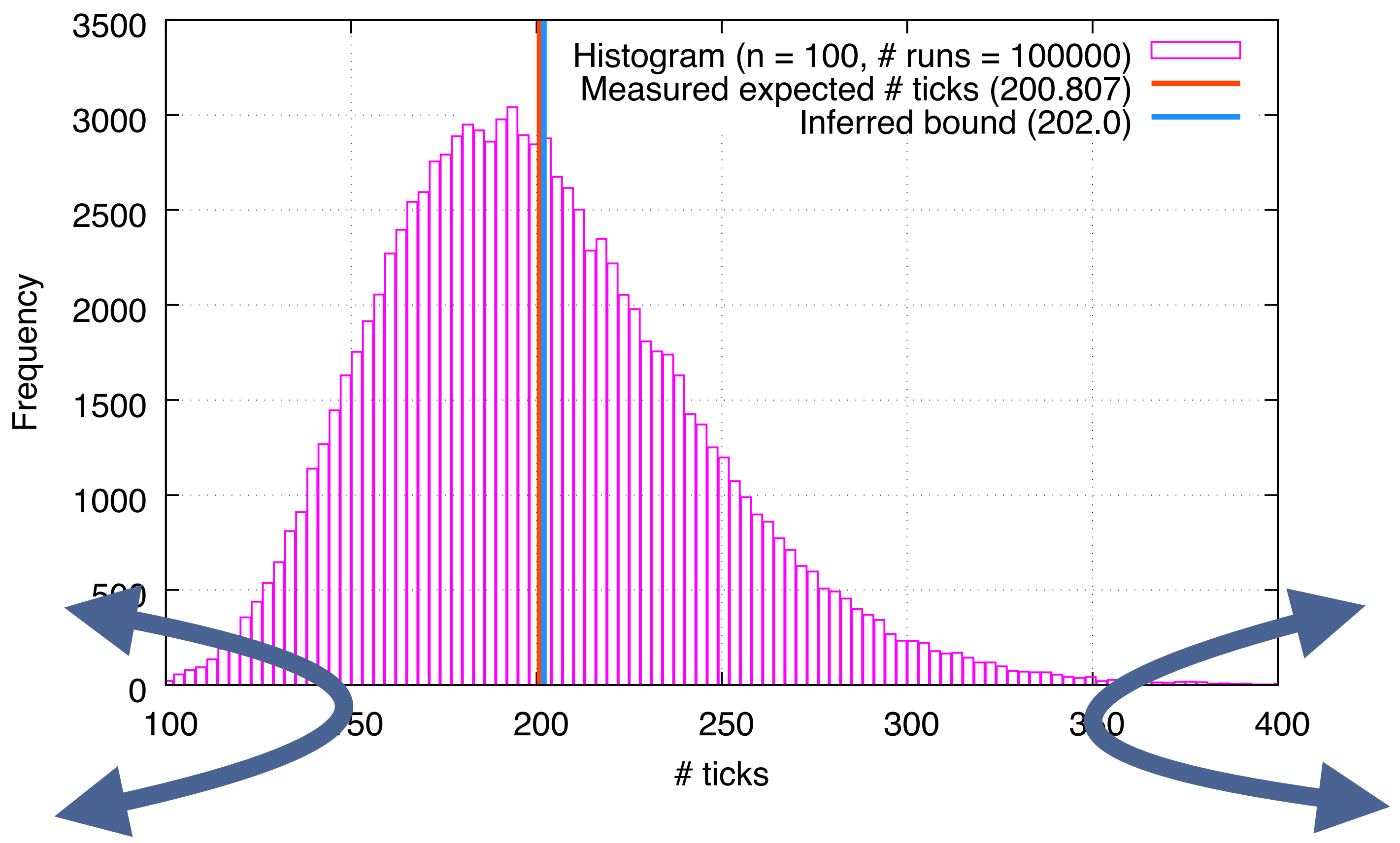


Red points are simulation-based values



Application: Tail-bound analysis



- Can be reduced to expected resource analysis using concentration inequalities (e.g., Markov and Chebyshev's inequalities)
- Assert that resource usage is bounded with a high probability
- Thus, they are good for analyzing safety properties of programs

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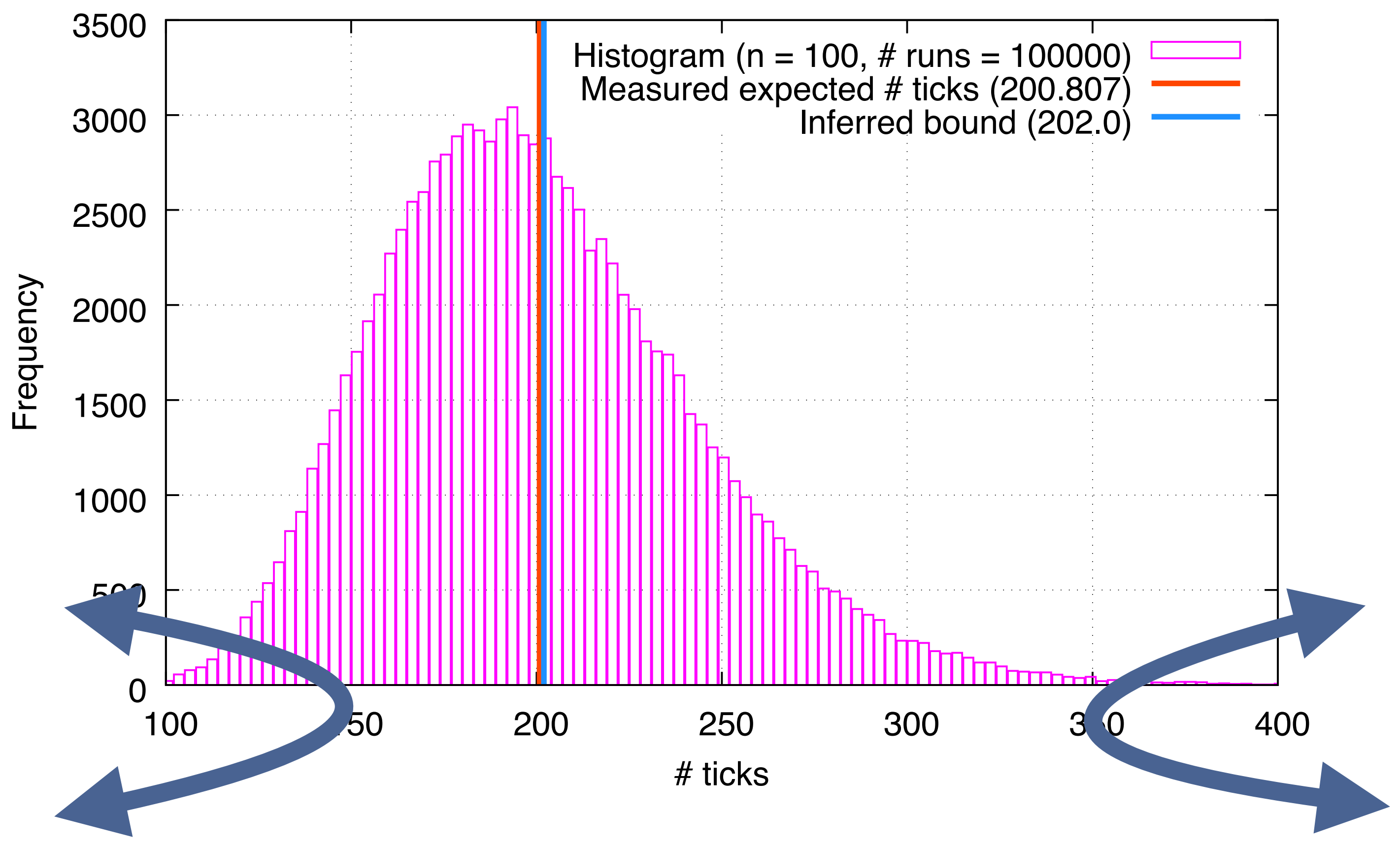

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Random walk example:

$$\mathbb{P}(t \geq 10|[0,n]|)$$

$$\leq \frac{\mathbb{E}(t)}{10|[0,n]|} \leq \frac{2|[0,n]|}{10|[0,n]|} = 0.2$$

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Future work

- Lower bounds on the expected resource usage
- Tail-bound analysis with Chebyshev’s inequality

Limitations

- Non-polynomial bounds
- Discrete distributions with finite domains